



The relative Brauer group of a cyclic cover of affine space

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ARTICLE INFO

Article history:

Received 12 March 2010

Received in revised form 2 June 2010

Available online 15 July 2010

Communicated by R. Parimala

MSC: 16K20; 16K50; 14F22

ABSTRACT

We study the finite-dimensional central division algebras over the rational function field in several variables over an algebraically closed field. We describe the division algebras that are split by the cyclic covering obtained by adjoining the n th root of a polynomial. The relative Brauer group is described in terms of the Picard group of the cyclic covering and its Galois group. Many examples are given and in most cases division algebras are presented that represent generators of the relative Brauer group.

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1. Introduction

Let k be an algebraically closed field of characteristic zero. Fix $m \geq 2$ and indeterminates x_1, \dots, x_m . Let $A = k[x_1, \dots, x_m]$ and f an irreducible polynomial in A defining a nonsingular hypersurface $Y = Z(f)$ in $\mathbb{A}^m = \text{Spec } k[x_1, \dots, x_m]$. Set $R = A[f^{-1}]$, $T = A[z]/(z^n - f)$, and $S = T[z^{-1}]$.

Let $K = k(x_1, x_2, \dots, x_m)$ be the field of rational functions on \mathbb{A}^m . Since the Brauer group is a functor, there is a homomorphism $B(R) \rightarrow B(S)$. The kernel is denoted by $B(S/R)$, and is the relative Brauer group mentioned in the title. We study the cohomology groups of S and T in order to compute $B(S/R)$. Moreover, we attempt to find the K -division algebras parametrized by the group $B(S/R)$. The main result of Section 2 is contained in Proposition 2.17 where $B(S/R)$ is described when $n = p$ is prime and T is simply connected (this hypothesis is explained below). In this case $B(S/R)$ is a finite \mathbb{Z}/p -module and the rank is determined by the Picard number of S . Along the way to this result, some auxiliary lemmas are proven. Section 3 consists of examples that illustrate the various aspects of Proposition 2.17. In Section 4, the methods of Section 2 are applied to some localizations of the rings defined above.

If we set $Z = Z(z) \subseteq \text{Spec } T$, then Z is a smooth divisor on the smooth variety $\text{Spec } T$. Since $Z \cong Y$ is a principal prime divisor we see that z is a prime element in T . Since Y is smooth we also see that $\text{Spec } T$ is a smooth divisor on \mathbb{A}^{m+1} . So T/A is the cyclic covering of degree n mentioned in the title. The covering T/A has branch locus Y . Since f is invertible in R , S/R is Galois with cyclic group G of order n . Let ω be a primitive n th root of unity in k and let $\sigma \in G$ be the generator for G whose action on S is induced by $z \mapsto \omega z$. The rings defined so far make up this subring diagram:

$$\begin{array}{ccccc}
 & & S = R[f^{1/n}] & & \\
 & \nearrow & & \nwarrow & \text{Galois} \\
 T = A[f^{1/n}] & & & & R = A[f^{-1}] \\
 & \nwarrow & & \nearrow & \\
 & & A & &
 \end{array} \tag{1}$$

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The corresponding diagram of varieties and morphisms is

$$\begin{array}{ccccc}
 \mathrm{Spec} S & \xrightarrow{\text{open}} & \mathrm{Spec} T & \xleftarrow{\text{closed}} & Z \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathrm{Spec} R & \xrightarrow{\text{open}} & \mathbb{A}^m & \xleftarrow{\text{closed}} & Y
 \end{array} \quad (2)$$

We suggest [15] as a standard reference for all unexplained terminology and notation. All sheaves and all cohomology are for the étale topology except when we utilize group cohomology. The ground field k has characteristic zero and is algebraically closed. For any scheme X of finite type over k , we denote by \mathbb{G}_m the sheaf of units and we write $X^* = H^0(X, \mathbb{G}_m)$ for the group of global units on X . We identify $\mathrm{Pic} X$, the Picard group of X , with $H^1(X, \mathbb{G}_m)$. The Brauer group is denoted $B(X)$ and there is a canonical embedding [8, I, (2.1), p. 51] $\delta : B(X) \rightarrow H^2(X, \mathbb{G}_m)$. By the Hoobler–Gabber Theorem (e.g. [11]), if X is the separated union of two affine schemes, the image of δ is equal to the torsion subgroup of $H^2(X, \mathbb{G}_m)$. For any abelian group M , and integer n , ${}_nM$ is the subgroup of M annihilated by n . The Kummer sequence [15, p. 66]

$$1 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \rightarrow 1 \quad (3)$$

is an exact sequence of sheaves for the étale topology on X , where μ_n is the constant sheaf on X of n th roots of unity. The long exact sequence of cohomology associated to (3) breaks up into the short exact sequences:

$$1 \rightarrow X^*/X^{*n} \rightarrow H^1(X, \mu_n) \rightarrow {}_n\mathrm{Pic} X \rightarrow 0 \quad (4)$$

$$0 \rightarrow \mathrm{Pic} X \otimes \mathbb{Z}/n \rightarrow H^2(X, \mu_n) \rightarrow {}_nB(X) \rightarrow 0 \quad (5)$$

$$0 \rightarrow H^{j-1}(X, \mathbb{G}_m) \otimes \mathbb{Z}/n \rightarrow H^j(X, \mu_n) \rightarrow {}_nH^j(X, \mathbb{G}_m) \rightarrow 0. \quad (6)$$

The sheaf of all roots of unity, $\mu = \bigcup_n \mu_n$, is non-canonically isomorphic to \mathbb{Q}/\mathbb{Z} . The group $H^1(X, \mu_n)$ is the set of isomorphism classes of Galois covers of X with group \mathbb{Z}/n [15, pp. 125–126]. Taking the limit over all n , we see that $H^1(X, \mathbb{Q}/\mathbb{Z})$ classifies the cyclic Galois covers of X . If X is a nonsingular variety, by [9, II, Proposition 1.4, p. 71] $H^j(X, \mathbb{G}_m)$ is a torsion group for $j \geq 2$. Taking limits in (6) yields

$$H^j(X, \mu) \cong H^j(X, \mathbb{G}_m) \quad (7)$$

for $j \geq 3$.

As before, $K = k(x_1, x_2, \dots, x_m)$ is the field of rational functions on \mathbb{A}^m . Introduce another variable x_0 and let $\mathbb{P}^m = \mathrm{Proj} k[x_0, x_1, \dots, x_m]$. We view \mathbb{A}^m as the Zariski open subset of \mathbb{P}^m defined by $x_0 \neq 0$. Since k is algebraically closed, $B(k) = 0$. By [2, Proposition 7.7], $B(\mathbb{A}^m) = B(k) = 0$. The diagram

$$\begin{array}{ccc}
 H^2(\mathbb{P}^m, \mathbb{G}_m) & \xrightarrow{\quad} & B(K) \\
 & \searrow \quad \swarrow & \\
 & H^2(\mathbb{A}^m, \mathbb{G}_m) &
 \end{array}$$

commutes and the maps are one-to-one by [9, II, Corollaire 1.10, p. 73]. But $B(\mathbb{A}^m) = H^2(\mathbb{A}^m, \mathbb{G}_m)$, so $B(\mathbb{P}^m) = H^2(\mathbb{P}^m, \mathbb{G}_m)$ is trivial.

Let Λ be a (finite-dimensional) central division algebra over K . So Λ represents a class $[\Lambda]$ in $B(K)$, the Brauer group of K . An important tool in studying K -central division algebras is the ramification map [1, Section 3]

$$B(K) \xrightarrow{\mathrm{ram}} \bigoplus_Z H^1(K(Z), \mathbb{Q}/\mathbb{Z}). \quad (8)$$

The direct sum in (8) is taken over all prime divisors Z on \mathbb{P}^m and the map ram is one-to-one since $B(\mathbb{P}^m) = 0$. The group $H^1(K(Z), \mathbb{Q}/\mathbb{Z})$ is the group of cyclic Galois extensions of the function field $K(Z)$ of Z . The map ram is also described in [19, Chapter 10]. The Brauer class $[\Lambda]$ is completely determined by the ramification data $\mathrm{ram}([\Lambda])$ in $\bigoplus H^1(K(Z), \mathbb{Q}/\mathbb{Z})$. The prime divisors Y_i where $\mathrm{ram}([\Lambda])$ is non-trivial make up the so-called ramification divisor $D = Y_1 \cup \dots \cup Y_v$ of Λ . By [1, (3.2), page 86] the group $B(\mathbb{P}^m - D)$ parametrizes those K -central division algebras Λ such that the ramification divisor of Λ is a subset of D . If $H = Z(x_0)$ is the hyperplane at infinity, then H is simply connected so $H^1(H, \mathbb{Q}/\mathbb{Z}) = 0$. It follows that $B(\mathbb{P}^m - H) = B(\mathbb{A}^m) = 0$. Since $R = H^0(\mathbb{A}^m - Y, \mathcal{O})$, $B(R)$ parametrizes those K -central division algebras that ramify on Y . We have the exact sequence [6, Corollary 1.4]

$$0 \rightarrow B(\mathbb{A}^m) \rightarrow B(R) \rightarrow H^1(Y, \mathbb{Q}/\mathbb{Z}) \rightarrow H^3(\mathbb{A}^m, \mu). \quad (9)$$

We already mentioned that the first group of (9) is trivial. By [15, Corollary VI.4.20], $0 = H^i(k, \mu_n) = H^i(\mathbb{A}^m, \mu_n)$ for all $n > 1$ and all $i > 0$. Taking the limit over n shows the last group in (9) is also trivial, hence $B(R) \cong H^1(Y, \mathbb{Q}/\mathbb{Z})$. That is, the Brauer group of R is isomorphic to the group of cyclic unramified Galois extensions of Y .

Let α, β be elements of K , $n \geq 2$ an integer, and ω a fixed n th root of unity in K . The symbol algebra $(\alpha, \beta)_n$ is the associative K -algebra generated by u, v subject to the relations $u^n = \alpha, v^n = \beta, uv = \omega vu$. It is a central simple K -algebra, hence represents a class in the Brauer group. The ramification divisor of the algebra $(\alpha, \beta)_n$ is contained in the set of zeros and poles of the function $\alpha\beta$ on \mathbb{P}^m .

2. Some lemmas

Throughout this section we continue to employ the same notation which was established in Section 1. In summary, k is an algebraically closed field of characteristic zero, $A = k[x_1, \dots, x_m]$ is the ring of polynomials in $m \geq 2$ variables over k , f is an irreducible polynomial in A , $R = A[f^{-1}]$, $T = A[z]/(z^n - f)$, and $S = T[f^{-1}]$, where $Y = Z(f)$ and $Z = Z(z)$ are nonsingular. Let $\mathbb{P}^m = \text{Proj } k[x_0, x_1, \dots, x_m]$, $H = Z(x_0)$ the hyperplane at infinity, $\mathbb{A}^m = \mathbb{P}^m - H$.

Lemma 2.1. *Let X denote the completion of $U = \text{Spec } T$ in \mathbb{P}^m . If X and $Y = X \cap H$ are both smooth, then $H^1(T, \mu_n) = 0$ for all $n \geq 2$.*

Proof. Since X is a smooth hypersurface in \mathbb{P}^m , $H^1(X, \mu_n) = 0$ by [5, p. 49]. Moreover if h denotes the class of the hyperplane section $Y = X \cap H$ in the Néron–Severi group $\text{NS}(X)$, then for all $n \geq 2$ the equation $h = nx$ has no solution for x . Associated to the closed immersion $Y \subseteq X$ are the cohomology groups with support on Y , $H_Y^j(X, F)$, for any sheaf F for the étale topology [15, pp. 91–95]. As in [15, pp. 247–248], there is a commutative diagram

$$\begin{array}{ccccccc} X^* & \longrightarrow & T^* & \longrightarrow & H_Y^1(X, \mathbb{G}_m) & \xrightarrow{\delta^1} & \text{Pic } X \longrightarrow \text{Pic } U \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X^* & \longrightarrow & T^* & \xrightarrow{\text{div}} & \mathbb{Z} \cdot Y & \longrightarrow & \text{Cl } X \longrightarrow \text{Cl } U \end{array} \quad (10)$$

where the top row is [15, Proposition III.1.25] and the bottom row is Nagata's Theorem (e.g. [7, Theorem 1.1]). Since X is nonsingular, $\text{Pic } X = \text{Cl } X$ [10, Corollary II.6.16] and the vertical arrows are all isomorphisms. Hence $H_Y^1(X, \mathbb{G}_m) \cong \mathbb{Z}$. By (4), $\text{Pic } X$ is torsion free. Since Y is not a principal divisor, δ^1 is one-to-one. Therefore $k^* = X^* = U^*$ and the sequence

$$0 \rightarrow \mathbb{Z} \cdot Y \rightarrow \text{Pic } X \rightarrow \text{Pic } U \rightarrow 0 \quad (11)$$

is exact. Combine (11) with the Kummer sequence (3) to get the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \cdot Y & \longrightarrow & \text{Pic } X & \longrightarrow & \text{Pic } U \longrightarrow 0 \\ & & \downarrow n & & \downarrow n & & \downarrow n \\ 0 & \longrightarrow & \mathbb{Z} \cdot Y & \longrightarrow & \text{Pic } X & \longrightarrow & \text{Pic } U \longrightarrow 0 \end{array} \quad (12)$$

in which the first two vertical maps are one-to-one. The Snake Lemma [18, Theorem 6.5] applied to (12) reduces to the exact sequence

$$0 \rightarrow H^1(U, \mu_n) \rightarrow \mathbb{Z}/n \xrightarrow{\delta} \text{Pic } X \otimes \mathbb{Z}/n \rightarrow \text{Pic } U \otimes \mathbb{Z}/n \rightarrow 0. \quad (13)$$

The group $H^1(U, \mu_n)$ is cyclic. The class of Y is not divisible in $\text{NS}(X)$, so δ is one-to-one. \square

From now on we assume $n = p^e$ is a positive power of a prime number p and $H^1(T, \mathbb{Z}/p^t) = 0$ for all $t > 0$. This last condition on T is true in the setting of Lemma 2.1, or by (4), if the group of units in T is equal to k^* and all torsion in $\text{Pic } T$ is relatively prime to p .

Lemma 2.2. *The following are true.*

(a) $H^1(R, \mathbb{Z}/p^\infty) \cong \mathbb{Z}/p^\infty$ and the sequence

$$0 \rightarrow \mathbb{Z}/n \rightarrow H^1(R, \mathbb{Z}/p^\infty) \rightarrow H^1(S, \mathbb{Z}/p^\infty) \rightarrow 0$$

is exact.

(b) $H^1(T, \mathbb{Z}/p^\infty) = 0$

(c) $\text{Pic } T = \text{Pic } S$ and all torsion is relatively prime to p .

(d) $k^* = T^*$ and $S^* = k^* \times \langle z \rangle$.

Proof. (a): Let $v = p^t$ for any $t \geq e$. The Gysin sequence for $Y \subseteq \mathbb{A}^m$ is [15, p. 244]

$$\dots \rightarrow H^i(\mathbb{A}^m, \mathbb{Z}/v) \rightarrow H^i(R, \mathbb{Z}/v) \xrightarrow{f_i} H^{i-1}(Y, \mathbb{Z}/v) \rightarrow \dots \quad (14)$$

For all $i > 0$ we have $H^i(\mathbb{A}^m, \mathbb{Z}/v) = 0$ and r_i is an isomorphism. When $i = 1$, the map r_1 is the ramification map for cyclic Galois extensions of R with group \mathbb{Z}/v . Therefore $H^1(R, \mathbb{Z}/v) \cong H^0(Y, \mathbb{Z}/v) = \mathbb{Z}/v$. A generator of $H^1(R, \mathbb{Z}/v)$ is the class corresponding to the R -Galois extension $R[f^{1/v}]$. Taking the limit over all $v = p^t$, $t \geq e$, shows that $H^1(R, \mathbb{Z}/p^\infty) \cong \mathbb{Z}/p^\infty$ and the cyclic extensions of R of degree v are all of the form $R[f^{j/v}]$. Consider the commutative diagram

$$\begin{array}{ccccccccc} T^* & \longrightarrow & S^* & \xrightarrow{\delta} & H^1_Z(T, \mathbb{G}_m) & \longrightarrow & \text{Pic } T & \longrightarrow & \text{Pic } S \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T^* & \longrightarrow & S^* & \xrightarrow{\text{div}} & \mathbb{Z} \cdot Z & \longrightarrow & \text{Cl } T & \longrightarrow & \text{Cl } S \end{array} \quad (15)$$

which is the counterpart of (10) for the closed immersion $Z \subseteq \text{Spec } T$. Since T is nonsingular, $\text{Pic } T = \text{Cl } T$ [10, Corollary II.6.16]. The vertical arrows are all isomorphisms. Since $Z = Z(z)$ is a principal divisor, $\text{Cl } T \cong \text{Cl } S$ and $H^1_Z(T, \mathbb{G}_m) \cong \mathbb{Z}$ is generated by $\delta(z)$. Combine the Kummer sequence (3) with the top row of (15) to get the commutative diagram

$$\begin{array}{ccccc} S^* & \xrightarrow{v} & S^* & \longrightarrow & H^1(S, \mu_v) \\ \downarrow \delta & & \downarrow \delta & & \downarrow \\ H^1_Z(T, \mathbb{G}_m) & \xrightarrow{v} & H^1_Z(T, \mathbb{G}_m) & \xrightarrow{\partial} & H^2_Z(T, \mu_v) \end{array}$$

The image of ∂ has order v . By Cohomological Purity [15, Theorem VI.5.1], there is an isomorphism $H^0(Z, \mathbb{Z}/v) \cong H^2_Z(T, \mu_v)$, so $H^2_Z(T, \mu_v)$ is cyclic of order v . Therefore ∂ is onto. The counterpart of (14) for $Z \subseteq \text{Spec } T$ starts with the low degree terms

$$0 \rightarrow H^1(T, \mathbb{Z}/v) \rightarrow H^1(S, \mathbb{Z}/v) \xrightarrow{r_1} H^0(Z, \mathbb{Z}/v) \rightarrow \cdots \quad (16)$$

By the above arguments, the ramification map r_1 sends the cyclic extension $S[z^{1/v}]$ to a generator of $H^0(Z, \mathbb{Z}/v)$. By hypothesis, $H^1(T, \mathbb{Z}/v) = 0$, so r_1 is an isomorphism. Since $S = R[f^{1/n}]$, in the commutative square

$$\begin{array}{ccc} H^1(R, \mathbb{Z}/v) & \xrightarrow{\cong} & H^0(Y, \mathbb{Z}/v) \\ \downarrow & & \downarrow \\ H^1(S, \mathbb{Z}/v) & \xrightarrow{\cong} & H^0(Z, \mathbb{Z}/v) \end{array} \quad (17)$$

the vertical maps are multiplication by $n = p^e$. The kernel of each vertical map is cyclic of order n and the sequence of (a) follows.

(b): By assumption, $H^1(T, \mathbb{Z}/p^t) = 0$. Take the limit over all t .

(c): We already saw that $\text{Pic } T = \text{Pic } S$. From the Kummer sequence (4) for T , it follows that the torsion in $\text{Pic } T$ is prime to p .

(d): From the second row of (15), $S^* \cong T^* \times \langle z \rangle$. From (4) it follows that T^* is p -divisible. To finish the proof it is enough to show that T^*/k^* is a finitely generated torsion free abelian group. If X is a projective completion of $U = \text{Spec } T$, and $\tilde{X} \rightarrow X$ is the normalization, then $\tilde{X} - U$ is a divisor. By Nagata's Theorem, T^*/k^* is isomorphic to a subgroup of a finitely generated torsion free abelian group. \square

Lemma 2.3. *There is an exact sequence*

$$0 \rightarrow B(S/R) \rightarrow H^1(Y, \mathbb{Z}/n) \rightarrow B(T) \rightarrow \frac{B(S)}{\text{im } (B(R))} \rightarrow H^1(Z, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}/n.$$

If $H^3(T, \mathbb{G}_m) = 0$, the last arrow is surjective. If R has dimension two (that is, if $m = 2$), $H^1(Z, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}/n = 0$.

Proof. Combine (9) and the counterpart of (9) for T and S . This gives the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B(R) & \longrightarrow & H^1(Y, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow n=p^e & & \\ B(T) & \xrightarrow{1\text{-to-1}} & B(S) & \longrightarrow & H^1(Z, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^3(T, \mathbb{G}_m) \end{array} \quad (18)$$

with exact rows. Since $Z \rightarrow Y$ is ramified with index n , the third vertical arrow in (18) is “multiplication by n ” [19, Theorem 10.4]. The Snake Lemma [18, p. 174] applied to (18) finishes the proof since $H^1(Y, \mathbb{Q}/\mathbb{Z}) = H^1(Z, \mathbb{Q}/\mathbb{Z})$ is divisible if Y is a nonsingular curve. \square

Question 2.4. In diagram (18) is it true that the map $H^1(Z, \mathbb{Q}/\mathbb{Z}) \rightarrow H^3(T, \mathbb{G}_m)$ is always the zero map? In other words is the last arrow in Lemma 2.3 always surjective?

Lemma 2.5. Let $G = \langle \sigma \rangle$ denote the Galois group of S/R . The following are true.

- (a) $H^i(G, S^*) = 0$ for $i > 0$.
 (b) $(\text{Pic } S)^G = 0$ and for all $i > 0$

$$H^{2i}(G, \text{Pic } S) = 0$$

and

$$H^{2i-1}(G, \text{Pic } S) \cong \frac{\text{Pic } S}{(\sigma - 1)\text{Pic } S}.$$

- (c) $B(S/R) \cong H^1(G, \text{Pic } S)$ and there is an exact sequence

$$0 \rightarrow H^1(G, \text{Pic } S) \rightarrow B(R) \rightarrow B(S)^G \rightarrow 0.$$

Proof. We use the Hochschild–Serre spectral sequence [15, p. 105]:

$$H^p(G, H^q(S, \mathbb{G}_m)) \Rightarrow H^{p+q}(R, \mathbb{G}_m).$$

Let $G = \langle \sigma \rangle$ where $\sigma : z \mapsto \omega z$, where ω is a fixed n th root of unity in k . The lower left-hand block of the spectral sequence looks like this:

$$\begin{array}{ccccccc} & B(S)^G & H^1(G, B(S)) & H^2(G, B(S)) & \cdots & & \\ & & & \searrow d_2^{1,2} & & & \\ E_2^{p,q}: & (\text{Pic } S)^G & H^1(G, \text{Pic } S) & H^2(G, \text{Pic } S) & H^3(G, \text{Pic } S) & & \\ & (S^*)^G & H^1(G, S^*) & H^2(G, S^*) & \cdots & & \end{array} \quad (19)$$

From Lemma 2.2(d), $S^* = k^* \times \langle z \rangle$ and the action of σ is $\sigma(uz^a) = u\omega^a z^a$ where $u \in k^*$, $a \in \mathbb{Z}$. We see that

$$\begin{aligned} (S^*)^G &= \{uz^a \in k^* \times \langle z \rangle \mid uz^a = u\omega^a z^a\} \\ &= \{u(z^n)^a\} \\ &= k^* \times \langle z^n \rangle. \end{aligned}$$

We follow the notation of [18, pp. 296–297]. In $\mathbb{Z}G$, set $D = \sigma - 1$ and $N = 1 + \sigma + \cdots + \sigma^{n-1}$. Compute

$$\begin{aligned} DS^* &= \{(-1 + \sigma)(uz^a) \mid uz^a \in k^* \times \langle z \rangle\} \\ &= \{u^{-1}z^{-a}u\omega^a z^a \mid uz^a \in k^* \times \langle z \rangle\} \\ &= \{\omega^a\} = \mu_n \end{aligned}$$

and

$$\begin{aligned} {}_N S^* &= \{uz^a \mid N(uz^a) = 1\} \\ &= \{uz^a \mid (1 + \sigma + \cdots + \sigma^{n-1})(uz^a) = 1\} \\ &= \{uz^a \mid (uz^a)(u\omega^a z^a)(u\omega^{2a} z^a) \cdots (u\omega^{(n-1)a} z^a) = 1\} \\ &= \{uz^a \mid u^n \omega^{a+2a+\cdots+(n-1)a} z^{na} = 1\}. \end{aligned}$$

But $u^n \omega^{a+2a+\cdots+(n-1)a} z^{na} = 1$ in $k^* \times \langle z \rangle$ if and only if $z^{na} = z^0$ (which implies $a = 0$) and $u^n = 1$ (which implies $u \in \mu_n$). Therefore ${}_N S^* = \mu_n$. Now check that

$$\begin{aligned} NS^* &= \{N(uz^a) \mid uz^a \in k^* \times \langle z \rangle\} \\ &= \{u^n (\omega^a)^{n(n-1)/2} z^{na} \mid uz^a \in k^* \times \langle z \rangle\} \\ &= \{uz^{na} \mid uz^a \in k^* \times \langle z \rangle\} \\ &= k^* \times \langle z^n \rangle. \end{aligned}$$

We find that

$$H^0(G, S^*) = (S^*)^G = k^* \times \langle z^n \rangle$$

and if $i \geq 1$

$$H^{2i-1}(G, S^*) = \frac{NS^*}{DS^*} = \frac{\mu_n}{\mu_n} = 0$$

and

$$H^{2i}(G, S^*) = \frac{(S^*)^G}{NS^*} = \frac{k^* \times \langle z^n \rangle}{k^* \times \langle z^n \rangle} = 0$$

which proves (a).

Consider the filtration $E^1 = E_0^1 \supseteq E_1^1 \supseteq 0$ of $E^1 = H^1(R, \mathbb{G}_m) = \text{Pic } R = 0$. This gives $E_1^1 = E_\infty^{1,0} = E_2^{1,0} = 0$ and $E_0^1 = E_\infty^{0,1} = E_2^{0,1} = (\text{Pic } S)^G = 0$. So

$$H^{2i}(G, \text{Pic } S) = \frac{(\text{Pic } S)^G}{N\text{Pic } S} = 0.$$

It follows that the norm map $N: \text{Pic } S \rightarrow \text{Pic } S$ is the zero map and $D = \sigma - 1$ is injective. By [18, Theorem 10.35], for each $i > 0$

$$H^{2i-1}(G, \text{Pic } S) \cong \frac{N\text{Pic } S}{D\text{Pic } S} \cong \frac{\text{Pic } S}{(\sigma - 1)\text{Pic } S}.$$

This proves (b).

Lastly consider the filtration $E^2 = E_0^2 \supseteq E_1^2 \supseteq E_2^2 \supseteq 0$ of $E^2 = H^2(R, \mathbb{G}_m) = B(R)$. This gives $E_2^2 = E_\infty^{2,0} = E_2^{2,0} = H^2(G, S^*) = 0$, $E_1^2 = E_\infty^{1,1} = E_2^{1,1} = H^1(G, \text{Pic } S)$ and $E_0^2/E_1^2 = E_\infty^{0,2} = E_2^{0,2} = (B(S))^G$. Therefore the sequence

$$0 \rightarrow H^1(G, \text{Pic } S) \rightarrow B(S)^G \rightarrow 0$$

is exact, proving (c). \square

Lemma 2.6. *If R has dimension two (that is, $m = 2$), then*

(a) $d_2^{1,2}: H^1(G, B(S)) \xrightarrow{\cong} H^3(G, \text{Pic } S)$ *is an isomorphism, and*

(b) $H^{2i}(G, B(S)) = 0$ *for all $i > 0$.*

Proof. For the affine surface R , the arguments used in the proof of [6, Lemma 0.1] show that for $j \geq 3$, $E^j = H^j(R, \mathbb{G}_m) = H^j(R, \mu)$ (the characteristic of k is always assumed to be zero). By [15, Theorem 7.2, p. 253] these groups are trivial. The filtration $0 = E^3 = E_0^3 \supseteq E_1^3 \supseteq E_2^3 \supseteq E_3^3$ yields $0 = E_1^3 = E_\infty^{1,2} = \ker d_2^{1,2}$. The filtration $0 = E^4 = E_0^4 \supseteq E_1^4 \supseteq E_2^4 \supseteq E_3^4 \supseteq \dots$ yields $0 = E_3^4 = E_\infty^{3,1} = \text{coker } d_2^{1,2}$. This proves (a). Since $0 = E_2^4 = E_\infty^{2,2} = E_2^{2,2} = H^2(G, B(S)) = (0)$, (b) follows. \square

Lemma 2.7. *If R has dimension two (that is, $m = 2$), then there is an exact sequence*

$$0 \rightarrow H^1(G, \text{Pic } S) \rightarrow {}_n B(R) \rightarrow {}_n B(S)^G \rightarrow H^1(G, \text{Pic } S) \rightarrow 0.$$

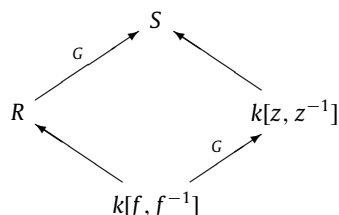
Proof. Since R is a nonsingular affine surface, $H^3(R, \mathbb{G}_m) = H^3(R, \mu) = H^3(R, \mu_n) = 0$. Therefore, by the Kummer sequence (6), $B(R)$ is divisible by n . Likewise, $B(S)$ is divisible by n . The exact sequence of Lemma 2.5(c) combined with “multiplication by n ” yields the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H^1(G, \text{Pic } S) & \longrightarrow & {}_n B(R) & \longrightarrow & {}_n B(S)^G \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H^1(G, \text{Pic } S) & \longrightarrow & B(R) & \longrightarrow & B(S)^G \longrightarrow 0 \\
 & & \downarrow n=0 & & \downarrow n & & \downarrow n \\
 0 & \longrightarrow & H^1(G, \text{Pic } S) & \longrightarrow & B(R) & \longrightarrow & B(S)^G \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H^1(G, \text{Pic } S) & & 0 & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

with exact rows and columns. Apply the Snake Lemma to finish the proof. \square

Lemma 2.8. *There is a natural isomorphism $H^2(R, \mathbb{Z}/n) \cong H^2(S, \mathbb{Z}/n)^G$.*

Proof. Consider the diagram of ring extensions



where $G = \langle \sigma \rangle$, $z^n = f$, $\sigma(z) = \omega z$, $\mu_n = \langle \omega \rangle$. Each of the extensions labelled with G is Galois with group G . Using Kummer theory and Lemma 2.2(c) one checks that

$$\begin{aligned}
 k^* \times \langle f \rangle &= k[f, f^{-1}]^* = R^*, \\
 \mu_n &= H^1(k[f, f^{-1}], \mathbb{Z}/n) = H^1(R, \mathbb{Z}/n), \\
 k^* \times \langle z \rangle &= k[z, z^{-1}]^* = R^*,
 \end{aligned}$$

and

$$\mu_n = H^1(k[z, z^{-1}], \mathbb{Z}/n) = H^1(S, \mathbb{Z}/n).$$

Consider the Hochschild–Serre spectral sequence

$$H^p(G, H^q(k[z, z^{-1}], \mathbb{Z}/n)) \Rightarrow H^{p+q}(k[f, f^{-1}], \mathbb{Z}/n).$$

The action of G on $H^0(k[z, z^{-1}], \mathbb{Z}/n)$ is trivial. Check as well that G acts trivially on $H^1(k[z, z^{-1}], \mathbb{Z}/n)$. Use [18, Corollary 10.36] to compute $E_2^{p,0} = H^p(G, \mathbb{Z}/n) = \mathbb{Z}/n$ and $E_2^{p,1} = H^p(G, \mu_n) = \mu_n$. Since $k[f, f^{-1}]$ has dimension one, by [15, Theorem 7.2, p. 253], $H^i(k[f, f^{-1}], \mathbb{Z}/n) = 0$ for all $i \geq 2$. Likewise $H^i(k[z, z^{-1}], \mathbb{Z}/n) = 0$ for all $i \geq 2$. Therefore $E_2^{p,2} = H^p(G, 0) = 0$. The lower left-hand corner looks like

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & & 0 & & \cdots \\
 E_2^{p,q}: & \mu_n & \xrightarrow{d_2^{0,1}} & \mu_n & \xrightarrow{d_2^{1,1}} & \mu_n & \xrightarrow{d_2^{2,1}} & \mu_n & \xrightarrow{\quad} & \cdots \\
 & \mathbb{Z}/n & & \mathbb{Z}/n & & \mathbb{Z}/n & & \mathbb{Z}/n & & \cdots
 \end{array} \tag{20}$$

First use the filtration for $E^2 = H^2(k[f, f^{-1}], \mathbb{Z}/n) = 0$, which is $0 = E_0^2 \supseteq E_1^2 \supseteq E_2^2 \supseteq 0$. It follows that $0 = E_2^2 = E_\infty^{2,0} = E_3^{2,0}$ hence $d_2^{0,1}$ is surjective hence $d_2^{0,1}$ is an isomorphism. Likewise $0 = E_1^2 = E_\infty^{1,1} = E_3^{1,1} = \ker d_2^{1,1}$ hence $d_2^{1,1}$ is an isomorphism. Next use the filtration for $E^4 = H^4(k[f, f^{-1}], \mathbb{Z}/n) = 0$, which is $0 = E_0^4 \supseteq \cdots \supseteq E_4^4$. It follows that $0 = E_4^4 = E_\infty^{4,0} = E_3^{4,0}$ hence $d_2^{2,1}$ is surjective hence $d_2^{2,1}$ is an isomorphism. By periodicity we know that $d_2^{p,1}$ is an isomorphism for each $p \geq 0$.

The diagram

$$\begin{array}{ccc}
 H^p(G, H^1(S, \mathbb{Z}/n)) & \xrightarrow{d_2^{p,1}} & H^{p+2}(G, H^0(S, \mathbb{Z}/n)) \\
 \downarrow = & & \downarrow = \\
 H^p(G, H^1(k[z, z^{-1}], \mathbb{Z}/n)) & \xrightarrow{\cong} & H^{p+2}(G, H^0(k[z, z^{-1}], \mathbb{Z}/n))
 \end{array} \tag{21}$$

commutes, so the top row is an isomorphism for all $p \geq 0$.

Next look at the Hochschild–Serre spectral sequence

$$H^p(G, H^q(S, \mathbb{Z}/n)) \Rightarrow H^{p+q}(R, \mathbb{Z}/n).$$

The lower left-hand terms look like

$$\begin{array}{ccccccc}
 E_2^{0,2} & & E_2^{1,2} & & E_2^{2,2} & & E_2^{3,2} & & \cdots \\
 E_2^{p,q}: & \mu_n & \xrightarrow{d_2^{0,2}} & \mu_n & \xrightarrow{d_2^{1,2}} & \mu_n & \xrightarrow{d_2^{2,2}} & \mu_n & \xrightarrow{d_2^{3,2}} & \cdots \\
 & \mathbb{Z}/n & & \mathbb{Z}/n & & \mathbb{Z}/n & & \mathbb{Z}/n & & \cdots
 \end{array} \tag{22}$$

From (21) we know $d_2^{p,1}$ is an isomorphism for $p \geq 0$. This means $E_3^{p,1} = 0$ for $p \geq 0$ and $E_3^{p,0} = 0$ for $p \geq 2$. This also means $d_2^{p,2} = 0$ and $E_3^{p,2} = E_2^{p,2}$ for $p \geq 0$. Therefore the third row has already converged. The next iteration of (22) becomes

$$\begin{array}{cccc}
 H^2(S, \mathbb{Z}/n)^G & E_2^{1,2} & E_2^{2,2} & \dots \\
 E_3^{p,q}: & 0 & 0 & 0 \quad \dots \\
 & \mathbb{Z}/n & \mathbb{Z}/n & 0 \quad \dots
 \end{array} \quad (23)$$

Look at the filtration for $E^2 = H^2(R, \mathbb{Z}/n)$. It is $E_0^2 = E_0^2 \supseteq E_1^2 \supseteq E_2^2 \supseteq 0$. From (23) we see that $E_2^2 = E_3^{2,0} = 0$ and $E_1^2 = E_3^{1,1} = 0$. So $E_0^2 = E_3^{0,2}$ which completes the proof. \square

Lemma 2.9. *There is an exact sequence*

$$0 \rightarrow B(S/R) \rightarrow \frac{\text{Pic } S}{n\text{Pic } S} \rightarrow \frac{H^2(S, \mu_n)}{H^2(S, \mu_n)^G} \rightarrow \frac{{}_nB(S)}{\text{im } {}_nB(R)} \rightarrow 0.$$

Proof. The diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^2(R, \mu_n) & \longrightarrow & {}_nB(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \frac{\text{Pic } S}{n\text{Pic } S} & \longrightarrow & H^2(S, \mu_n) & \longrightarrow & {}_nB(S) \longrightarrow 0
 \end{array}$$

commutes and the rows are exact. Apply the Snake Lemma and Lemma 2.8. \square

Lemma 2.10. *If R has dimension two (that is, $m = 2$), then $H^p(G, H^2(S, \mathbb{Z}/n)) = 0$ for all $p > 0$.*

Proof. By [15, Theorem 7.2, p. 253], $H^i(S, \mathbb{Z}/n) = 0$ for all $i > 2$. Then (23) becomes

$$\begin{array}{cccc}
 0 & 0 & 0 & \dots \\
 E_3^{p,q}: & H^2(S, \mathbb{Z}/n)^G & E_3^{1,2} & E_3^{2,2} \quad \dots \\
 & 0 & 0 & 0 \quad \dots \\
 & \mathbb{Z}/n & \mathbb{Z}/n & 0 \quad \dots
 \end{array}$$

Since $0 = H^3(R, \mathbb{Z}/n) = E_0^3 \supseteq E_1^3$ we see $0 = E_1^3 = E_3^{1,2} = E_2^{1,2} = H^1(G, H^2(S, \mathbb{Z}/n))$. Likewise $0 = H^4(R, \mathbb{Z}/n) = E_0^4 \supseteq E_2^4 = E_3^{2,2} = E_2^{2,2} = H^2(G, H^2(S, \mathbb{Z}/n))$. By periodicity $H^p(G, H^2(S, \mathbb{Z}/n)) = 0$ for all $p > 0$. \square

Lemma 2.11. *The following are true.*

(a) *There is a natural isomorphism:*

$$\left(\frac{\text{Pic } S}{n\text{Pic } S} \right)^G \cong H^1(G, \text{Pic } S).$$

(b) *There are natural isomorphisms for all $i \geq 1$:*

$$H^{2i-1}(G, \text{Pic } S) \cong H^{2i-1}\left(G, \frac{\text{Pic } S}{n\text{Pic } S}\right).$$

(c) *There are natural isomorphisms for all $i \geq 0$:*

$$H^{2i}\left(G, \frac{\text{Pic } S}{n\text{Pic } S}\right) \cong H^{2i+1}(G, \text{Pic } S).$$

Proof. By Lemma 2.2(c) there is a short exact sequence

$$0 \rightarrow \text{Pic } S \xrightarrow{n} \text{Pic } S \rightarrow \frac{\text{Pic } S}{n\text{Pic } S} \rightarrow 0.$$

By Lemma 2.5(b), $(\text{Pic } S)^G = 0$ and $H^{2i}(G, \text{Pic } S) = 0$ for $i > 0$. Therefore the long exact sequence of cohomology groups becomes

$$\begin{aligned} 0 \rightarrow \left(\frac{\text{Pic } S}{n\text{Pic } S} \right)^G &\rightarrow H^1(G, \text{Pic } S) \xrightarrow{n=0} \\ H^1(G, \text{Pic } S) &\rightarrow H^1\left(G, \frac{\text{Pic } S}{n\text{Pic } S}\right) \rightarrow 0 \\ 0 \rightarrow H^2\left(G, \frac{\text{Pic } S}{n\text{Pic } S}\right) &\rightarrow H^3(G, \text{Pic } S) \xrightarrow{n=0} \dots \end{aligned}$$

Multiplication by n is the zero map, since all groups are n -torsion. The proof follows from this sequence. \square

Proposition 2.12. *There are natural isomorphisms*

$$\frac{\text{Pic } S}{(\sigma - 1)\text{Pic } S} \cong B(S/R) \cong \left(\frac{\text{Pic } S}{n\text{Pic } S} \right)^G.$$

Proof. Follows from Lemma 2.5(c) and Lemma 2.11(a). \square

Remark 2.13. The exact sequence of Lemma 2.9 combines with the conclusion of Proposition 2.12 to yield the short exact sequence

$$0 \rightarrow \frac{\frac{\text{Pic } S}{n\text{Pic } S}}{\left(\frac{\text{Pic } S}{n\text{Pic } S} \right)^G} \rightarrow \frac{H^2(S, \mu_n)}{H^2(S, \mu_n)^G} \rightarrow \frac{nB(S)}{\text{im } nB(R)} \rightarrow 0.$$

Lemma 2.14. *There are bases for the finitely generated \mathbb{Z} -module part of $\text{Pic } S$ such that the matrix for $\sigma - 1: \text{Pic } S \rightarrow \text{Pic } S$ is the diagonal $\{d_1, d_2, \dots, d_r\}$, each d_i divides d_{i+1} and $d_1 | d_2 | \dots | d_r | n$. If $n = 2$, then $d_1 = d_2 = \dots = d_r = 2$ and*

$$\frac{\text{Pic } S}{(\sigma - 1)\text{Pic } S} = \frac{\text{Pic } S}{2\text{Pic } S}$$

Proof. Use the basis theorem for finitely generated abelian groups. Lemma 2.5(b) and Lemma 2.9 combine to show that

$$\frac{\text{Pic } S}{(\sigma - 1)\text{Pic } S} \rightarrow \frac{\text{Pic } S}{n\text{Pic } S}$$

is injective, so $d_r | n$. By Lemma 2.5(b), 1 is not an eigenvalue of σ . It follows that if $n = 2$, then $d_1 = d_2 = \dots = d_r = 2$. \square

Assume $n = 2$ and let $\pi: \text{Spec } S \rightarrow \text{Spec } R$ be the projection morphism. Following [14, p. 80], since S/R is Galois of degree two, the sequence of constant sheaves on $\text{Spec } R$

$$1 \rightarrow \mu_2 \rightarrow \pi_* \mu_2 \xrightarrow{\text{tr}} \mu_2 \rightarrow 1$$

is exact, hence gives rise to a long exact sequence of cohomology for R . Since π is finite, π_* is exact, so $R^q \pi_* \mu_2 = 0$ for $q > 0$ and the spectral sequence degenerates into $H^i(R, \pi_* \mu_2) = H^i(S, \mu_2)$ for all $i \geq 0$. Since $\text{Spec } S$ is connected $H^0(R, \pi_* \mu_2) = \mu_2$. The terms of low degree in the long exact sequence are

$$\begin{aligned} \mu_2 &\xrightarrow{\partial^0} H^1(R, \mu_2) \xrightarrow{\text{res}^1} H^1(S, \mu_2) \xrightarrow{\text{cor}^1} H^1(R, \mu_2) \\ &\xrightarrow{\partial^1} H^2(R, \mu_2) \xrightarrow{\text{res}^2} H^2(S, \mu_2) \xrightarrow{\text{cor}^2} H^2(R, \mu_2) \xrightarrow{\partial^2} H^3(R, \mu_2) \end{aligned} \quad (24)$$

It also follows from Lemma 2.8 that res^2 is injective, hence $\partial^1 = 0$. Lemma 2.2(a) implies ∂^0 is an isomorphism and cor^1 is an isomorphism.

Lemma 2.15. *Assume $n = 2$. The sequence*

$$0 \rightarrow H^2(R, \mu_2) \xrightarrow{\text{res}} H^2(S, \mu_2) \xrightarrow{\text{cor}} H^2(R, \mu_2)$$

is exact. If R has dimension two (that is, $m = 2$) then the map cor is surjective.

Proof. To prove the second claim, we note that when R is an affine surface the cohomology $H^i(R, \mu_2)$ vanishes for $i > 2$. \square

Lemma 2.16. Assume $n = 2$ and R has dimension two (that is, $m = 2$). Then

$$B(S/R) = \frac{\text{Pic } S}{2\text{Pic } S} = \left(\frac{\text{Pic } S}{2\text{Pic } S} \right)^G = \frac{\text{Pic } S}{(\sigma - 1)\text{Pic } S}$$

and in the notation of Lemma 2.14, the diagonalization of $\sigma - 1: \text{Pic } S \rightarrow \text{Pic } S$ is $\{2, 2, \dots, 2\}$.

Proof. From Proposition 2.12 it suffices to show $B(S/R) \cong \text{Pic } S \otimes \mathbb{Z}/2$. There is a commutative diagram

$$\begin{array}{ccccccc} & & H^2(R, \mu_2) & \xrightarrow{\cong} & {}_2B(R) & & \\ & & \downarrow \text{res} & & \downarrow \text{res} & & \\ 0 & \longrightarrow & \text{Pic } S \otimes \mathbb{Z}/2 & \longrightarrow & H^2(S, \mu_2) & \longrightarrow & {}_2B(S) \longrightarrow 0 \\ & & \downarrow \text{cor} & & \downarrow \text{cor} & & \\ & & H^2(R, \mu_2) & \xrightarrow{\cong} & {}_2B(R) & & \end{array}$$

with exact rows from (5). The vertical map ${}_2B(S) \rightarrow {}_2B(R)$ is corestriction and is defined in [14]. A diagram chase completes the proof. Moreover this proves that

$${}_2B(R) \xrightarrow{\text{res}} {}_2B(S) \xrightarrow{\text{cor}} {}_2B(R) \rightarrow 0$$

is exact. \square

In the notation of Lemma 2.14, assume $n = p$ is prime. Then $(\sigma - 1)$ can be diagonalized into the form $\{1, \dots, 1, p, \dots, p\}$. Let s count the multiplicity of 1 in this diagonal and t the multiplicity of p . By Lemma 2.14, if $p = 2$ then $s = 0$. What can we say about s and t for $p > 2$? Look at the matrix over \mathbb{Q} for σ acting on $\text{Pic } S \otimes \mathbb{Q}$. Put it into rational canonical form with blocks on the diagonal. Each block will be the companion matrix for the cyclotomic polynomial $\Phi_p(x) = (x^p - 1)/(x - 1) = x^{p-1} + \dots + x + 1$. So the blocks are of size $p - 1$ -by- $p - 1$. That is, there exists a basis for $\text{Pic } S \otimes \mathbb{Q}$ for which the matrix of σ is blocks of size $p - 1$. Then $p - 1$ divides the rank of the torsion free part of $\text{Pic } S$. It follows from Proposition 2.17 below that $s = (p - 2)t$. By [4, p. 508] $\text{Pic } S$ decomposes into a direct sum of irreducible $\mathbb{Z}G$ -modules

$$(A_1, a_1) \oplus \dots \oplus (A_r, a_r) \oplus A_{r+1} \oplus \dots \oplus A_m \oplus Y$$

where Y is a free \mathbb{Z} -module and G acts trivially on Y , A_1, \dots, A_m are $\mathbb{Z}[\omega]$ -ideals in $\mathbb{Q}[\omega]$ and $a_i \in A_i$, $a_i \notin (\omega - 1)A_i$. Since $(\text{Pic } S)^G = 0$, we see that $Y = 0$. Since (A_i, a_i) has \mathbb{Z} -rank p , it follows that $(A_i, a_i) \otimes \mathbb{Q}$ has \mathbb{Q} -dimension p . Since 1 is not an eigenvalue for σ over \mathbb{Q} , we see that (A_i, a_i) does not appear as a direct summand of $\text{Pic } S$. We conclude that $r = 0$ so

$$\text{Pic } S \cong A_1 \oplus \dots \oplus A_t$$

as $\mathbb{Z}G$ -modules. Each A_i is a $\mathbb{Z}[\omega]$ -ideal in $\mathbb{Q}[\omega]$. The \mathbb{Z} -rank of A_i is $p - 1$, the action of σ on A_i is multiplication by ω . That is, $\sigma(a) = \omega a$ for all $a \in A_i$. It follows that $A_i^G = 0$.

Proposition 2.17. Let $n = p$ be prime. There is an isomorphism of $\mathbb{Z}G$ -modules $\text{Pic } S \cong A_1 \oplus \dots \oplus A_t$ (modulo torsion subgroups) where A_1, \dots, A_t are $\mathbb{Z}[\omega]$ -ideals in $\mathbb{Q}[\omega]$. It follows that the finitely generated torsion free subgroup of $\text{Pic } S$ has \mathbb{Z} -rank equal to $(p - 1)t$ and there are isomorphisms

$$B(S/R) \cong \frac{\text{Pic } S}{(1 - \sigma)\text{Pic } S} \cong (\text{Pic } S \otimes \mathbb{Z}/p)^G \cong H^1(G, \text{Pic } S) \cong (\mathbb{Z}/p)^{(t)}.$$

Proof. It suffices to prove this when $t = 1$. View $A = A_1$ as a free \mathbb{Z} -module and $A \otimes \mathbb{Q}$ as a \mathbb{Q} -vector space. With respect to some basis, the matrix of $\sigma \in \text{Mat}_{p-1}(\mathbb{Q})$ is the companion matrix of the cyclotomic polynomial $\Phi_p(x)$ of order p :

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & -1 & -1 & \dots & -1 & -1 \end{bmatrix}$$

and $1 - \sigma$ has the $(p - 1)$ -by- $(p - 1)$ matrix

$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & -1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 2 \end{bmatrix}.$$

It is easy enough to check that the determinant of Q is p . In the notation of Lemma 2.14, the invariants of the matrix Q are: $r = p - 1$, $d_1 = \dots = d_{p-2} = 1$, $d_{p-1} = p$. This means $A/(1 - \sigma)A$ is cyclic of order p . \square

3. Some examples

We apply some of the results of Section 2 and provide some examples that illustrate various aspects of Proposition 2.17.

3.1. Plane rational ramification curve

Example 3.1. This is a two-dimensional example which illustrates the results of Proposition 2.17. Let $A = k[x, y]$, $f = xy + 1$, $R = A[f^{-1}]$, $T = A[z]/(z^n - f)$ and $S = T[z^{-1}]$. Let $Y = Z(f) \subseteq \mathbb{A}^2$. Let $G = \langle \sigma \rangle$ be the Galois group of S/R where the action of σ is $\sigma(z) = \omega z$, ω being a primitive n th root of unity in k . This notation agrees with the notation of Section 1.

Since $k[x, y]/(xy + 1) \cong k[y, y^{-1}]$, we see that Y is nonsingular, affine, rational, factorial, and has two points at infinity. Then $H^0(Y, \mathcal{O}^*) = k^* \times \langle y \rangle$ and $H^1(Y, \mathbb{Z}/n) \cong H^0(Y, \mathcal{O}^*/\mathcal{O}^{*n}) \cong \mathbb{Z}/n$. By (9) it follows that ${}_n B(R) \cong H^1(Y, \mathbb{Z}/n) \cong \mathbb{Z}/n$. Construct the symbol algebra $\Lambda = (y, f)_n$ over $R[y^{-1}]$. It follows that Λ is non-split and extends to an Azumaya R -algebra. Therefore Λ is a generator for ${}_n B(R)$. Since Λ is split by S , we see that $B(S/R) = {}_n B(R)$.

Since $T/(z) \cong A/(f)$, we see that (z) is a principal prime ideal. Therefore $\text{Pic } T \cong \text{Pic } S$. Note that T is rational since there is an isomorphism

$$T[y^{-1}] \xrightarrow{\cong} k[y, y^{-1}, z]$$

defined by sending $y \mapsto y$ and $x \mapsto (z^n - 1)y^{-1}$. Then $T[y^{-1}]^* = H^0(k[y, y^{-1}, z], \mathcal{O}^*) = k^* \times \langle y \rangle$. Since $T[y^{-1}]$ is factorial, $\text{Pic } T = \text{Cl } T$ is generated by the minimal primes containing y . The primes of height one containing y are $\{P_i = (y, z - \omega^i)\}_{i=1}^n$. In this set-up, Nagata's Theorem (for example, see [7, Theorem 1.1]) gives an exact sequence

$$1 \rightarrow T^* \rightarrow T[y^{-1}]^* \rightarrow \bigoplus_{i=1}^n \mathbb{Z}P_i \rightarrow \text{Cl } T \rightarrow 0.$$

So $\text{Cl } T$ is generated by P_1, \dots, P_n subject to the relation $(y) \sim 0$. In the local ring T_{P_i} we have

$$z - \omega^i = yx \frac{z - \omega^i}{z^n - 1}$$

so y is a local parameter for the divisor P_i . The divisor of y is $(y) = P_1 + \dots + P_n$. Therefore $\text{Cl } T \cong \mathbb{Z}^{(n-1)}$ is a free abelian group of rank $n - 1$ with basis $\{P_1, \dots, P_{n-1}\}$.

To see how this fits into the context of Proposition 2.17, say n is prime. We have shown that $t = 1$ and $\text{Pic } S \cong A_1$, where A_1 is an irreducible $\mathbb{Z}G$ -module of \mathbb{Z} -rank $n - 1$. Now assume $n > 1$ is arbitrary and continue.

The action of σ on $\text{Cl } T$ is given by

$$\begin{aligned} \sigma(P_1) &= P_n = -P_1 - \dots - P_{n-1} \\ \sigma(P_2) &= P_1 \\ \vdots &= \vdots \\ \sigma(P_{n-1}) &= P_{n-2}. \end{aligned}$$

The matrix of $1 - \sigma$ is

$$Q = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & -1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 & 1 & -1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

This is an $(n - 1)$ -by- $(n - 1)$ matrix. Use induction and expand along the last row to see that the determinant of Q is n . Check that the quotient group $\text{Pic } T / (1 - \sigma)\text{Pic } T$ is a cyclic group of order n . This proves that

$$\frac{\text{Pic } S}{(1 - \sigma)\text{Pic } S} \cong \left(\frac{\text{Pic } S}{n\text{Pic } S} \right)^G \cong H^1(G, \text{Pic } S) \cong \mathbb{Z}/n.$$

As we have already seen, these groups are isomorphic to $B(S/R) = {}_n B(R)$. When n is a prime, these computations agree with the results predicted by Proposition 2.17.

Example 3.2. This example starts with A as in Example 3.1, but f is taken to be a more general polynomial. Set $f = f(x, y) = xg(y) + 1$ where

$$g(y) = \prod_{i=1}^d (y - \lambda_i)^{e_i}$$

is non-constant and has $d > 0$ distinct roots $\lambda_1, \dots, \lambda_d$. Let $n > 1$ be arbitrary, $R = A[f^{-1}]$, $T = A[z]/(z^n - f)$ and $S = T[z^{-1}]$. Let $Y = Z(f) \subseteq \mathbb{A}^2$. Let $G = \langle \sigma \rangle$ be the Galois group of S/R where the action of σ is $\sigma(z) = \omega z$, ω being a primitive n th root of unity in k . By (9) it follows that $B(R) \cong H^1(Y, \mathbb{Q}/\mathbb{Z})$. Note that Y is nonsingular, factorial, affine, rational, and intersects the line at infinity in two points. At the $x = 0$ point, the intersection is normal. At the $y = 0$ point, there are d distinct tangent directions. Then $H^0(Y, \mathcal{O}^*) = k^* \times \prod_{i=1}^d \langle y - \lambda_i \rangle$ and $H^1(Y, \mathbb{Z}/n) \cong H^0(Y, \mathcal{O}^*/\mathcal{O}^{*n}) \cong (\mathbb{Z}/n)^{(d)}$. Then ${}_n B(R) \cong (\mathbb{Z}/n)^{(d)}$.

Construct d symbol algebras $A_i = (y - \lambda_i, f)_n$ over $k(x, y)$. A computation involving the methods of [6, Section 2] shows that A_i is non-split and extends to an Azumaya R -algebra and $\{A_1, \dots, A_d\}$ is a \mathbb{Z}/n -basis for ${}_n B(R)$. We have shown that $B(S/R) = {}_n B(R)$.

As in Lemma 2.2, (z) is a principal prime ideal and $\text{Pic } T \cong \text{Pic } S$. Note that T is rational since there is an isomorphism

$$T[g(y)^{-1}] \xrightarrow{\cong} k[y, g(y)^{-1}, z]$$

defined by sending $y \mapsto y$ and $x \mapsto (z^n - 1)g(y)^{-1}$. Therefore

$$B(T[g(y)^{-1}]) \cong B(k[y, g(y)^{-1}, z]) \cong B(k[y, g(y)^{-1}]) = 0.$$

This implies $B(T) = 0$ so

$$B(S/R) = {}_n B(R) = H^1(Y, \mathbb{Z}/n) = (\mathbb{Z}/n)^d.$$

Since $T[g(y)^{-1}]$ is factorial, $\text{Pic } T = \text{Cl } T$ is generated by the minimal primes containing $g(y)$. As in Example 3.1, $\text{Pic } T$ is generated by the minimal prime ideals $P_{i,j} = (y - \lambda_i, z - \omega^j)$ where $i = 1, \dots, d$ and $j = 1, \dots, n - 1$. A local parameter for $T_{P_{i,j}}$ is $y - \lambda_i$. Following the Example 3.1 argument, we see that $\text{Pic } T$ decomposes into d direct summands, each of which is an irreducible $\mathbb{Z}G$ module. The quotient group $\text{Pic } T / (1 - \sigma)\text{Pic } T$ is a direct sum of d cyclic groups, each of order n . This proves that

$$B(S/R) = {}_n B(R) \cong \frac{\text{Pic } S}{(1 - \sigma)\text{Pic } S} \cong \left(\frac{\text{Pic } S}{n\text{Pic } S} \right)^G \cong H^1(G, \text{Pic } S) \cong (\mathbb{Z}/n)^d.$$

If n is prime, then in the notation of Proposition 2.17, we have confirmed that $t = d$ and $\text{Pic } S$ has \mathbb{Z} -rank equal to $(n - 1)t$.

Example 3.3. Here is a three-dimensional example which illustrates the results of Proposition 2.17. We extend Example 3.1 to three dimensions. Let $A = k[x_1, x_2, x_3]$, $f = x_1 x_2 x_3 + 1$, $R = A[f^{-1}]$, $T = A[z]/(z^n - f)$ and $S = T[z^{-1}]$. Let $Y = Z(f) \subseteq \mathbb{A}^3$. Let $G = \langle \sigma \rangle$ be the Galois group of S/R where the action of σ is $\sigma(z) = \omega z$, ω being a primitive n th root of unity in k . This notation agrees with the notation of Section 1.

First compute the action of σ on the Picard group of T . Note that T is rational since there is an isomorphism

$$T[x_1^{-1}, x_2^{-1}] \xrightarrow{\cong} k[x_1, x_1^{-1}, x_2, x_2^{-1}, z]$$

defined by $x_3 \mapsto (z^n - 1)x_1^{-1}x_2^{-1}$. Then

$$T[x_1^{-1}, x_2^{-1}]^* = H^0(k[x_1, x_1^{-1}, x_2, x_2^{-1}, z], \mathcal{O}^*) = k^* \times \langle x_1 \rangle \times \langle x_2 \rangle.$$

Since $T[x_1^{-1}, x_2^{-1}]$ is factorial, $\text{Pic } T = \text{Cl } T$ is generated by the minimal primes containing x_1 and x_2 . For $i = 1$ or $i = 2$, there are exactly n primes of height one that contain x_i , namely $\{P_{i,j} = (x_i, z - \omega^j)\}_{j=1}^n$. In this set-up, Nagata's Theorem (for example, see [7, Theorem 1.1]) gives an exact sequence

$$1 \rightarrow T^* \rightarrow T[x_1^{-1}, x_2^{-1}]^* \rightarrow \bigoplus_{i=1}^2 \bigoplus_{j=1}^n \mathbb{Z}P_{i,j} \rightarrow \text{Cl } T \rightarrow 0.$$

So $\text{Cl } T$ is generated by the $2n$ prime divisors $P_{1,1}, \dots, P_{2,n}$ subject to the two relations $\text{div}(x_1) \sim 0$ and $\text{div}(x_2) \sim 0$. For either $i = 1$ or $i = 2$, in the local principal ideal ring $T_{P_{i,j}}$, we have

$$z - \omega^i = x_1 x_2 x_3 \frac{z - \omega^i}{z^n - 1}$$

so x_i is a local parameter for the divisor $P_{i,j}$. The divisor of x_i is $\text{div}(x_i) = P_{i,1} + \dots + P_{i,n}$. Therefore $\text{Cl } T \cong \mathbb{Z}^{(n-1)} \oplus \mathbb{Z}^{(n-1)}$ is a free abelian group of rank $2(n - 1)$ with basis $\{P_{1,1}, \dots, P_{1,n-1}, P_{2,1}, \dots, P_{2,n-1}\}$. The action of σ on $\text{Cl } T$ is given by

$$\begin{aligned} \sigma(P_{i,1}) &= P_{i,n} - P_{i,1} - \dots - P_{i,n-1} \\ \sigma(P_{i,2}) &= P_{i,1} \\ &\vdots \\ \sigma(P_{i,n-1}) &= P_{i,n-2}. \end{aligned}$$

As in Example 3.1 we find that

$$\begin{aligned} \frac{\text{Pic } S}{(1 - \sigma)\text{Pic } S} &\cong \left(\frac{\text{Pic } S}{n\text{Pic } S} \right)^G \\ &\cong H^1(G, \text{Pic } S) \\ &\cong \mathbb{Z}/n \oplus \mathbb{Z}/n. \end{aligned} \quad (25)$$

On Y we can set $x_3 = -1/(x_1x_2)$ and eliminate x_3 . Then $k[x_1, x_2, x_3]/(x_1x_2x_3 + 1) \cong k[x_1, x_2, x_1^{-1}, x_2^{-1}]$. We see that Y is nonsingular, affine, rational, factorial, and is isomorphic to the two-dimensional torus. Then $H^0(Y, \mathcal{O}^*) = k^* \times \langle x_1 \rangle \times \langle x_2 \rangle$ and $H^1(Y, \mathbb{Z}/n) \cong H^0(Y, \mathcal{O}^*/\mathcal{O}^{*n}) \cong \mathbb{Z}/n \oplus \mathbb{Z}/n$. By (9) it follows that

$${}_nB(R) \cong H^1(Y, \mathbb{Z}/n) \cong \mathbb{Z}/n \oplus \mathbb{Z}/n.$$

Construct the two symbol algebras $\Lambda_1 = (x_1, f)_n$, $\Lambda_2 = (x_2, f)_n$ over $R[x_1^{-1}, x_2^{-1}]$. It follows that each Λ_i is non-split and extends to an Azumaya R -algebra. Therefore Λ_1, Λ_2 are generators for ${}_nB(R)$. Since each Λ_i is split by S , we see that $B(S/R) = {}_nB(R)$. We have shown that the groups in (25) are isomorphic to ${}_nB(R) = B(S/R)$. When n is prime, these computations agree with the results predicted by Proposition 2.17.

It is straightforward to extend Example 3.3 to $m > 3$ dimensions, in which case we find that $B(S/R) \cong \mathbb{Z}/n^{(m-1)}$.

3.2. Nonsingular cubic ramification curve

In this section we investigate the relative Brauer group of a triple cover of \mathbb{P}^2 which ramifies along a nonsingular cubic curve. We try to set up our notation so that it agrees as much as possible with that from Section 1.

Let $g = g(x_0, x_1, x_2)$ be an irreducible cubic form in $k[x_0, x_1, x_2]$ defining $\bar{Y} = Z(g)$, a nonsingular cubic in \mathbb{P}^2 . Let $X = Z(x_3^3 - g(x_0, x_1, x_2))$. Then X is a nonsingular cubic surface in \mathbb{P}^3 . Let $\pi: X \rightarrow \mathbb{P}^2$ be the triple covering. Then π ramifies along \bar{Y} . Set $\bar{Z} = \pi^{-1}(\bar{Y})$. Then $\pi: X - \bar{Z} \rightarrow \mathbb{P}^2 - \bar{Y}$ is cyclic Galois with group $G = \langle \sigma \rangle$ where the action of σ is induced by $\sigma x_3 = \omega x_3$, ω being a primitive cube root of unity in k . Since \bar{Y} has genus one, the subgroup of \bar{Y} consisting of elements satisfying $a^3 = 1$ is a group of order nine. These are the nine flex points on \bar{Y} . Call them P_1, \dots, P_9 and let the corresponding tangent lines be called L_1, \dots, L_9 . Let $l_i \in k[x_0, x_1, x_2]$ be the linear form defining L_i . Since P_i is a flex, $g = a^3 + l_i b$ where $a, b \in k[x_0, x_1, x_2]$ are forms, a being linear and b quadratic. Therefore over $k[x_0, x_1, x_2, x_3]$ the equation for X is

$$x_3^3 - g = x_3^3 - a^3 + l_i b = 0$$

and it follows that $\pi^{-1}(L_i) = L_{i1} + L_{i2} + L_{i3}$ consists of three lines on X passing through the point $\pi^{-1}(P_i)$. The action of σ permutes the three lines L_{i1}, L_{i2}, L_{i3} cyclically. The twenty seven lines on X [16, Section 8D] are $\{L_{ij} \mid i = 1, \dots, 9; j = 1, 2, 3\}$.

Now \bar{Z} is a hyperplane section of X in \mathbb{P}^3 . In $\text{Cl } X = \text{Pic } X$, we have $\bar{Z} \sim \pi^* \bar{Y}$ is fixed by σ . We construct a \mathbb{Z} -basis for $\text{Pic}(X - \bar{Z})$. Start with L_{11} , one of the lines lying over L_1 . Then σL_{11} also lies over L_1 . Set $C_1 = L_{11}$ and $C_2 = \sigma L_{11}$. Following [10, V.4.10.1] we can do this two more times. That is, from the nine flex points on \bar{Y} , pick three that are not collinear. Re-label if necessary, and call the three points P_1, P_2, P_3 . Over each of the tangents for these three points, pick one of the three lines L_{ij} . Now set $C_1 = L_{11}, C_2 = \sigma L_{11}, C_3 = L_{21}, C_4 = \sigma L_{21}, C_5 = L_{31}$ and $C_6 = \sigma L_{31}$. The divisors C_1, \dots, C_6 are a \mathbb{Z} -module basis for $\text{Pic}(X - \bar{Z})$. Define submodules of $\text{Pic}(X - \bar{Z})$ by $A_1 = \mathbb{Z}C_1 + \mathbb{Z}C_2, A_2 = \mathbb{Z}C_3 + \mathbb{Z}C_4, A_3 = \mathbb{Z}C_5 + \mathbb{Z}C_6$. Each A_i is an irreducible $\mathbb{Z}G$ -submodule and $\text{Pic}(X - \bar{Z}) = A_1 \oplus A_2 \oplus A_3$. Then

$$\frac{A_i}{(1 - \sigma)A_i} \cong \mathbb{Z}/3$$

and

$$\frac{\text{Pic}(X - \bar{Z})}{(1 - \sigma)\text{Pic}(X - \bar{Z})} \cong \mathbb{Z}/3^{(3)}.$$

Now localize to the affine case. Let $H = Z(x_0)$ denote the “hyperplane at infinity” in either \mathbb{P}^3 or \mathbb{P}^2 . Set $\mathbb{A}^2 = \mathbb{P}^2 - H$, $Y = \bar{Y} - H$ and $R = H^0(\mathbb{A}^2 - Y, \mathcal{O})$. Let $W = H \cap X$ and $S = H^0(X - \bar{Z} - W, \mathcal{O})$. We have the Galois extension S/R which is in the same context as Section 1. We mention that any division algebra of exponent d over the rational function field of \mathbb{P}^2 with ramification divisor \bar{Y} is cyclic. This has been proved in [13] for arbitrary d , and [20] for odd d .

3.2.1. Case 1: \bar{Y} is in general position

Assume that \bar{Y} is in general position. So \bar{Y} intersects H in three distinct points, none of which is a flex. So W is an irreducible cubic curve on X . The exact sequence

$$1 \rightarrow k^* \rightarrow S^* \rightarrow \mathbb{Z} \cdot W \rightarrow \text{Pic}(X - \bar{Z}) \rightarrow \text{Pic } S \rightarrow 0 \quad (26)$$

and the fact that $S^* = k^* \times \langle x_3 \rangle$ by Lemma 2.2(4) shows that $\text{Pic}(X - \bar{Z}) = \text{Pic } S \cong \mathbb{Z}^{(6)}$. The previous computations together with Proposition 2.17 show

$$B(S/R) \cong \frac{\text{Pic } S}{(1-\sigma)\text{Pic } S} \cong \left(\frac{\text{Pic } S}{3\text{Pic } S} \right)^G \cong H^1(G, \text{Pic } S) \cong \mathbb{Z}/3^{(3)}. \quad (27)$$

Computations show that ${}_3B(R) = H^1(Y, \mathbb{Z}/3) \cong \mathbb{Z}/3^{(4)}$ so there is an exact sequence

$$0 \rightarrow B(S/R) \rightarrow {}_3B(R) \rightarrow \mathbb{Z}/3 \rightarrow 0.$$

Next we try to find the R -algebras in $B(S/R)$. As above, let $l_i = 0$ be the equation for the tangent L_i to \bar{Y} at the flex P_i . If we dehomogenize with respect to x_0 then l_i defines a polynomial function in $K = k(x_1, x_2)$. For each $i \neq j$ construct a symbol algebra $A_{ij} = (l_i/l_j, f)_3$ over K . Using the methods of [6, Section 2] we see that each A_i has ramification divisor Y hence A_i represents an element of $B(S/R)$. The function l_i/l_j on Y has divisor $(l_i/l_j) = 3P_i - 3P_j$. Hence $\text{ram } A_{ij}$ is in the subgroup of $H^1(\bar{Y}, \mathbb{Q}/\mathbb{Z})$ corresponding to the nine flex points on the elliptic curve \bar{Y} . We see that the algebras $A_{i,j}$ generate a subgroup of $B(S/R)$ of order nine and index three. By (27) there is another algebra class in $B(S/R)$ which we have not yet described. Can it be written as a symbol in the form $(\lambda, f)_3$?

3.2.2. Case 2: \bar{Y} is not in general position

There are several possible arrangements in this case. We look only at the case where H is a tangent to \bar{Y} . Assume that $\bar{Y} \cdot H = 3P_1$. Then $W = L_{11} + L_{12} + L_{13}$ and (26) becomes

$$1 \rightarrow k^* \rightarrow S^* \rightarrow \bigoplus_{j=1}^3 \mathbb{Z} \cdot L_{1j} \rightarrow \text{Pic}(X - \bar{Z}) \rightarrow \text{Pic } S \rightarrow 0. \quad (28)$$

From (28) and the computations above, we see that $\text{Pic } S \cong A_2 \oplus A_3$ and (27) becomes

$$B(S/R) \cong \frac{\text{Pic } S}{(1-\sigma)\text{Pic } S} \cong \left(\frac{\text{Pic } S}{3\text{Pic } S} \right)^G \cong H^1(G, \text{Pic } S) \cong \mathbb{Z}/3^{(2)}. \quad (29)$$

Computations show that ${}_3B(R) = H^1(Y, \mathbb{Z}/3) \cong \mathbb{Z}/3^{(2)}$. In this case $B(S/R) = {}_3B(R)$ and the symbol algebras $A_{ij} = (l_i/l_j, f)_3$ defined above are generators for this group.

3.3. Ramification along a quartic Fermat curve

In this section we investigate the relative Brauer group of a cyclic cover of degree four of \mathbb{P}^2 which ramifies along a nonsingular Fermat quartic \bar{Y} . We obtain a lower bound for the size of the group $B(S/R)$. We mention that any division algebra of exponent d over the rational function field of \mathbb{P}^2 with ramification divisor \bar{Y} is cyclic [12].

Let $g = x_0^4 + x_1^4 + x_2^4$ and $\bar{Y} = Z(g)$ the nonsingular Fermat quartic in \mathbb{P}^2 . Let $X = Z(x_3^4 - x_0^4 - x_1^4 - x_2^4)$. Then X is a $K3$ surface in \mathbb{P}^3 . Let $\pi: X \rightarrow \mathbb{P}^2$ be the projection. Then π ramifies along \bar{Y} . Set $\bar{Z} = \pi^{-1}(\bar{Y})$. Then $\pi: X - \bar{Z} \rightarrow \mathbb{P}^2 - \bar{Y}$ is cyclic Galois with group $G = \langle \sigma \rangle$ where the action of σ is induced by $\sigma x_3 = \omega x_3$, ω being a primitive fourth root of unity in k . Since \bar{Y} has genus three, the subgroup of $\text{Pic } \bar{Y}$ consisting of elements satisfying $a^4 = 1$ is a $\mathbb{Z}/4$ -module of rank six. There are twelve flex points on \bar{Y} given by intersecting \bar{Y} with the three coordinate axes $x_0 x_1 x_2 = 0$. Label them P_1, \dots, P_{12} and let the corresponding tangent lines be called L_1, \dots, L_{12} . Let $l_i \in k[x_0, x_1, x_2]$ be the linear form defining L_i . Since P_i is a flex, $g = a^4 + l_i b$ where $a, b \in k[x_0, x_1, x_2]$ are forms, a being linear and b cubic. Therefore over $k[x_0, x_1, x_2, x_3]$ the equation for X is

$$x_3^4 - g = x_3^4 - a^4 - l_i b = 0$$

and it follows that $\pi^{-1}(L_i) = L_{i1} + L_{i2} + L_{i3} + L_{i4}$ consists of four lines on X passing through the point $\pi^{-1}(P_i)$. The action of σ permutes the four lines $L_{i1}, L_{i2}, L_{i3}, L_{i4}$ cyclically. The forty eight lines on X are $\{L_{ij} \mid i = 1, \dots, 12; j = 1, \dots, 4\}$.

Since X is an exceptional $K3$ surface (see for example [3]), the Picard number of X is 20. As in Section 3.2 we construct the submodule of $\text{Pic}(X - \bar{Z})$ which is spanned by the prime divisors L_{ij} . Pick six of the points P_i (re-label so they are P_1, \dots, P_6) in such a way that the 18 divisors $\{L_{i1}, \sigma L_{i1}, \sigma^2 L_{i1}\}_{i=1}^6$ span a $\mathbb{Z}G$ -module direct summand of $\text{Pic}(X - \bar{Z})$ of \mathbb{Z} -rank 18. We refer to this submodule as M . So M is a direct sum of six $\mathbb{Z}G$ -submodules $A_i = \mathbb{Z} \cdot L_{i1} \oplus \mathbb{Z} \cdot \sigma L_{i1} \oplus \mathbb{Z} \cdot \sigma^2 L_{i1}$ where $i = 1, \dots, 6$. Then we see that $\text{Pic } X / (1 - \sigma)\text{Pic } X$ contains a submodule isomorphic to $M / (1 - \sigma)M \cong \mathbb{Z}/4^{(6)}$.

Now localize to the affine case. Let $H = Z(x_0)$ denote the “hyperplane at infinity” in either \mathbb{P}^3 or \mathbb{P}^2 . Set $\mathbb{A}^2 = \mathbb{P}^2 - H$, $Y = \bar{Y} - H$ and $R = H^0(\mathbb{A}^2 - Y, \mathcal{O})$. Let $W = H \cap X$ and $S = H^0(X - \bar{Z} - W, \mathcal{O})$. We have the Galois extension S/R which is in the same context as Section 1. So W is a Fermat quartic curve on X . The exact sequence (26) and the fact that $S^* = k^* \times \langle x_3 \rangle$ by Lemma 2.2(d) shows that $\text{Pic}(X - \bar{Z}) = \text{Pic } S$. The previous computations together with Proposition 2.12 show $B(S/R)$ contains a submodule isomorphic to $\mathbb{Z}/4^{(6)}$.

Since \bar{Y} has genus three, computations show that $H^1(\bar{Y}, \mathbb{Z}/4) \cong \mathbb{Z}/4^{(6)}$. Hence

$${}_4B(\mathbb{P}^2 - \bar{Y}) \cong H^1(\bar{Y}, \mathbb{Z}/4) \cong \mathbb{Z}/4^{(6)}$$

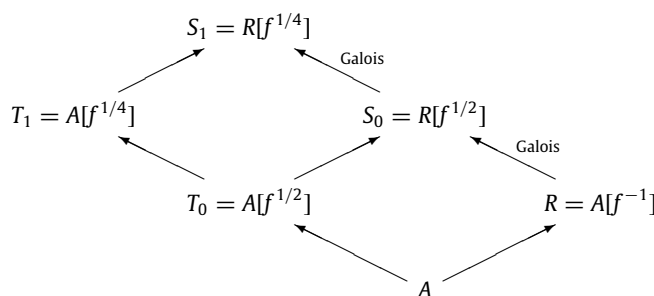
and

$${}_4B(R) \cong H^1(Y, \mathbb{Z}/4) \cong \oplus H^1(\bar{Y}, \mathbb{Z}/4) \oplus \mathbb{Z}/4^{(3)}.$$

We try to find the R -algebras in $B(S/R)$. As above, let $l_i = 0$ be the equation for the tangent L_i to \bar{Y} at the flex P_i . If we dehomogenize with respect to x_0 then l_i defines a polynomial function in $K = k(x_1, x_2)$. For each $i \neq j$ construct a symbol algebra $\Lambda_{ij} = (l_i/l_j, f)_3$ over K . Using the methods of [6, Section 2] we see that each Λ_i has ramification divisor Y hence Λ_i represents an element of $B(S/R)$. The function l_i/l_j on Y has divisor $(l_i/l_j) = 3P_i - 3P_j$. Hence $\text{ram } \Lambda_{ij}$ is the element of $H^1(\bar{Y}, \mathbb{Q}/\mathbb{Z})$ corresponding to the point on the Jacobian of \bar{Y} defined by the divisor $\{P_i - P_j\}$. By the computations of [17], it follows that the algebras $\Lambda_{i,j}$ generate a proper subgroup of $B(S/R)$ isomorphic to $\mathbb{Z}/4^{(5)} \oplus \mathbb{Z}/2$. In fact, each Λ_{ij} extends to the open $\mathbb{P}^2 - \bar{Y}$. We have shown that the kernel of $\pi^*: B(\mathbb{P}^2 - \bar{Y}) \rightarrow B(X - \bar{Z})$ contains a subgroup isomorphic to $\mathbb{Z}/4^{(5)} \oplus \mathbb{Z}/2$.

3.4. Smooth elliptic ramification curve

Let $f = x_2^2 - x_1(x_1 - 1)(x_1 - \lambda)$ where λ is different from 0 and 1. Then $Y = Z(f)$ is a smooth elliptic curve on \mathbb{A}^2 . In this example we look at both the quadratic and quartic cyclic covers of \mathbb{A}^2 that ramify along Y . Since Y is a hyperelliptic curve, according to [13] every division algebra with ramification divisor Y is in fact cyclic. This example shows that the Brauer classes of exponent four are split by adjoining $f^{1/4}$. Here are the rings we consider: $A = k[x_1, x_2]$, $R = A[f^{-1}]$, $T_0 = A[f^{1/2}]$, $S_0 = T_0[f^{-1}]$, $T_1 = A[f^{1/4}]$, $S_1 = T_1[f^{-1}]$. We have the following subring diagram.



First, we illustrate Proposition 2.17 for the quadratic extension S_0/R . We prove that the following are true.

$$B(T_0) = 0 \quad (30)$$

$$B(S_0/R) = {}_2B(R) = \mathbb{Z}/2^{(2)} \quad (31)$$

$$\text{Pic } T_0 = \text{Pic } S_0 = \mathbb{Z}^{(2)} \quad (32)$$

$$\frac{\text{Pic } S_0}{(1 - \sigma)\text{Pic } S_0} = \mathbb{Z}/2^{(2)}. \quad (33)$$

From (9) we know $B(R) \cong H^1(Y, \mathbb{Q}/\mathbb{Z})$ and since $Y = Z(f)$ is a nonsingular elliptic curve with one point at infinity, $H^1(Y, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}^{(2)}$. Also S_0 splits the ramification of every element in ${}_2B(R)$. Then (31) follows from (30).

To prove (30), start with the isomorphism

$$T_0 = \frac{k[x_1, x_2, z_0]}{z_0^2 - x_2^2 + x_1(x_1 - 1)(x_1 - \lambda)} \xrightarrow{\cong} T'_0 = \frac{k[x_1, u, v]}{uv + x_1(x_1 - 1)(x_1 - \lambda)}$$

defined by $u = z_0 - x_2$ and $v = z_0 + x_2$. Invert v and eliminate u to see that

$$T'_0[v^{-1}] = k[x_1, v, v^{-1}].$$

Now $B(T'_0) \rightarrow B(T'_0[v^{-1}])$ is injective because T_0 is nonsingular. Since $B(T'_0[v^{-1}]) = B(k[x_1, v, v^{-1}]) = 0$, it follows that $B(T_0) = 0$ which proves (30).

Now we prove (32). Recall that $\text{Pic } T_0 = \text{Pic } S_0$ by Lemma 2.2(3). The class group of T'_0 is generated by the prime ideals of height one containing v . There are three such ideals, namely $I_1 = (v, x_1)$, $I_2 = (v, x_1 - 1)$ and $I_3 = (v, x_1 - \lambda)$. Then $\text{div } v = I_1 + I_2 + I_3$. Nagata's Theorem (e.g. [7, Theorem 1.1]) gives the exact sequence

$$\mathbb{Z} \xrightarrow{1 \mapsto \text{div } v} \mathbb{Z} \cdot I_1 \oplus \mathbb{Z} \cdot I_2 \oplus \mathbb{Z} \cdot I_3 \rightarrow \text{Cl } T'_0 \rightarrow 0$$

which proves that $\text{Pic } T_0 = \text{Cl } T_0 \cong \mathbb{Z}^2$. Moreover this proves that $\text{Cl } T_0$ is equal to the internal direct sum $\mathbb{Z} \cdot I_1 \oplus \mathbb{Z} \cdot I_2$. This completes (32).

To prove (33) we determine the action of σ on $\text{Pic } S_0$. The three minimal primes containing u are $J_1 = (u, x_1)$, $J_2 = (u, x_1 - 1)$ and $J_3 = (u, x_1 - \lambda)$ and we see that $\sigma(I_i) = J_i$. Since $\text{div}(x_1) = I_1 + J_1$ and $\text{div}(x_1 - 1) = I_2 + J_2$, the action of σ is $\sigma(I_1) = -I_1$ and $\sigma(I_2) = -I_2$. We have shown $(1 - \sigma)\text{Pic } S_0 = 2\text{Pic } S_0$ and (33) follows.

Now we prove that the following are true for the cyclic extension S_1/R of degree 4.

$$B(T_1) = 0 \quad (34)$$

$$B(S_1/S_0) = {}_2B(S_0) = \mathbb{Z}/2^{(2)} \quad (35)$$

$$B(S_1/R) = {}_4B(R) = \mathbb{Z}/4^{(2)} \quad (36)$$

$$\text{Pic } T_1 = \text{Pic } S_1 = \mathbb{Z}^{(6)}. \quad (37)$$

Since $B(T_0) = 0$ by (30), it follows that $B(S_0) = H^1(Y, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}^{(2)}$. As S_1 splits the ramification of every element in ${}_2B(S_0)$, as well as the ramification of every element in ${}_4B(R)$, (35) and (36) both follow from (34).

To prove (34), identify T_1 with $T_0[z_0^{1/2}]$ and consider the isomorphism

$$T_1 = \frac{k[x_1, x_2, z_0, z_1]}{(z_0^2 - x_2^2 + x_1(x_1 - 1)(x_1 - \lambda), z_1^2 - z_0)} \xrightarrow{\cong} \frac{k[x_1, u, v, z_1]}{(uv + x_1(x_1 - 1)(x_1 - \lambda), 2z_1^2 - v - u)}$$

defined by $u = z_0 - x_2$ and $v = z_0 + x_2$. Eliminate u and identify

$$T_1 = \frac{k[x_1, v, z_1]}{(2vz_1^2 - v^2 + x_1(x_1 - 1)(x_1 - \lambda))}.$$

Introduce a fourth homogeneous variable w . Set

$$\Gamma = \frac{k[w, x_1, v, z_1]}{(2vz_1^2 - wv^2 + x_1(x_1 - w)(x_1 - \lambda w))}$$

and view $\text{Spec } T_1$ as the open subset of $\text{Proj } \Gamma$ where $w \neq 1$. The “curve at infinity” on $\text{Proj } \Gamma$ is

$$H = Z(w) = \text{Proj} \frac{k[x_1, v, z_1]}{x_1^3 + 2vz_1^2}$$

which is a plane cuspidal cubic curve. Then $Z(w)$ is simply connected, so $H^1(H, \mathbb{Q}/\mathbb{Z}) = 0$. The exact sequence

$$0 \rightarrow B(\text{Proj } \Gamma) \rightarrow B(\text{Spec } T_1) \rightarrow H^1(H, \mathbb{Q}/\mathbb{Z}) \rightarrow 0$$

shows that it is enough to prove $B(\text{Proj } \Gamma) = 0$. It is routine to check that $\text{Proj } \Gamma$ is nonsingular using the Jacobian criterion. Since $\text{Proj } \Gamma$ is a nonsingular cubic in \mathbb{P}^3 , it is a rational surface with trivial Brauer group and (34) follows. We know as well (for example, [16, Section 8D] or [10, V.4]) that the Picard group of $\text{Proj } \Gamma$ is isomorphic to \mathbb{Z}^7 and a hyperplane section generates a direct summand. The sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \text{Pic}(\text{Proj } \Gamma) \rightarrow \text{Pic}(\text{Spec } T_1) \rightarrow 0$$

is split exact, where the first map sends a generator for \mathbb{Z} to the class of H , the hyperplane section. So $\text{Pic } S_1 = \text{Pic } T_1 \cong \mathbb{Z}^6$ which is (37).

If we apply Proposition 2.12 to the Galois extension S_1/R we get

$$\frac{\text{Pic } S_1}{(1 - \sigma)\text{Pic } S_1} = \left(\frac{\text{Pic } S_1}{4\text{Pic } S_1} \right)^G = \mathbb{Z}/4^{(2)}$$

where $G = \langle \sigma \rangle$ is the Galois group.

3.5. Smooth quartic ramification curve

Let $g = x_2^4 + x_1(x_1 - 1)(x_1 - \lambda)$ where λ is different from 0 and 1. Then $Y = Z(g)$ is a smooth quartic curve on \mathbb{A}^2 . In this example we look at the double cover of \mathbb{A}^2 that ramifies along Y . Here are the rings we consider: $A = k[x_1, x_2]$, $R = A[g^{-1}]$, $T = A[g^{1/2}]$, $S = T[g^{-1}]$. We have the following subring diagram.

$$\begin{array}{ccc}
 & S = T[g^{-1}] & \\
 & \swarrow \quad \searrow & \\
 T = \frac{k[x_1, x_2, z]}{(z^2 - x_2^4 - x_1(x_1 - 1)(x_1 - \lambda))} & & R = A[g^{-1}] \\
 & \nwarrow \quad \nearrow & \\
 & A = k[x_1, x_2] &
 \end{array}$$

Galois

We check that the following are true.

$$B(T) = 0 \quad (38)$$

$$B(S/R) = {}_2B(R) = \mathbb{Z}/2^{(6)} \quad (39)$$

$$\text{Pic } T = \text{Pic } S = \mathbb{Z}^{(6)}. \quad (40)$$

First of all, notice that the ring T is isomorphic to the ring T_1 of Section 3.4, so (38) and (40) are true. From (9) we know $B(R) \cong H^1(Y, \mathbb{Q}/\mathbb{Z})$ and since $Y = Z(f)$ is a nonsingular quartic curve with one point at infinity, $H^1(Y, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}^{(6)}$. Also S splits the ramification of every element in ${}_2B(R)$, so (39) is true. Since T is an affine surface with trivial Brauer group, $B(S) \cong H^1(Y, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}^{(6)}$ by [6]. According to [12] every division algebra over $k(x_1, x_2)$ with ramification divisor Y is cyclic and has index equal to the exponent.

If we apply Proposition 2.17 to the Galois extension S/R we get

$$\frac{\text{Pic } S}{(1 - \sigma)\text{Pic } S} = \frac{\text{Pic } S}{2\text{Pic } S} = \mathbb{Z}/2^{(6)}$$

where $G = \langle \sigma \rangle$ is the Galois group.

4. Application to a localization

We apply techniques from Section 2 to compute the relative Brauer group in a context obtained by localizing the rings of Section 1. In this section, the rings R and S are localizations of their counterparts of Section 1. Let f_1, \dots, f_t be distinct linear polynomials in $A = k[x_1, \dots, x_m]$ and $R = A[f_1^{-1}, \dots, f_t^{-1}]$. Let $T = A[f_1^{1/n}]$ and $S = R[f_1^{1/n}]$. Then S/R is Galois with cyclic group $G = \langle \sigma \rangle$. If we let $z = f_1^{1/n}$, then $\sigma(z) = \omega z$, $\omega^n = 1$. Since f_1 is linear, $T \cong A$ and $\text{Pic } T = 0$ implies $\text{Pic } S = 0$. The rings defined so far make up this subring diagram.

$$\begin{array}{ccccc} & & S = R[f_1^{1/n}] & & \\ & \nearrow & & \nwarrow & \text{Galois} \\ T = A[f_1^{1/n}] & & & & R = A[f_1^{-1}, \dots, f_t^{-1}] \\ & \nwarrow & & \nearrow & \\ & & A & & \end{array} \quad (41)$$

The corresponding diagram of varieties and morphisms is

$$\begin{array}{ccccc} \text{Spec } S & \xrightarrow{\text{open}} & \text{Spec } T & \xleftarrow{\text{closed}} & Z \\ \pi \downarrow & & \pi \downarrow & & \pi \downarrow \\ \text{Spec } R & \xrightarrow{\text{open}} & \mathbb{A}^m & \xleftarrow{\text{closed}} & Y \end{array} \quad (42)$$

where Y_i denotes the hyperplane in \mathbb{A}^m defined by $f_i = 0$, $Y = Y_1 + \dots + Y_t$ and $Z = \pi^{-1}(Y)$.

Consider $\pi^{-1}(Y_i)$. There are three cases to consider. First, if $i = 1$, then Y_1 is the ramification divisor of π and $\pi^{-1}(Y_1)$ consists of one irreducible component which is isomorphic to \mathbb{A}^{m-1} . For the other two cases, assume $i = 2$. The ramification locus of the morphism $\pi: \pi^{-1}(Y_2) \rightarrow Y_2$ is the closed set $Y_1 \cap Y_2$. The second case occurs if $Y_1 \cap Y_2$ is non-empty. Then $Y_1 \cap Y_2 \cong \mathbb{A}^{m-2}$ and the ideal (f_1, f_2) in A is prime of height two. In T the ideal (f_2) is prime of height one and $\pi^{-1}(Y_2) \cong \mathbb{A}^{m-1}$ is irreducible. In the third case, $Y_{12} = Y_1 \cap Y_2$ is empty and the ideal (f_1, f_2) in A is the unit ideal. So $f_2 = f_1 + \alpha_2$, for some α_2 in k^* . In T , f_2 factors into n distinct factors and $\pi^{-1}(Y_2) = Y_{21} + Y_{22} + \dots + Y_{2n}$ has n irreducible components. Each irreducible component Y_{2i} is isomorphic to \mathbb{A}^{m-1} .

In light of the above computations, we re-label f_2, \dots, f_t so that Y_2, \dots, Y_p satisfy:

$$\emptyset = Y_1 \cap Y_2 = Y_1 \cap Y_3 = \dots = Y_1 \cap Y_p$$

and Y_{p+1}, \dots, Y_t satisfy

$$\emptyset \neq Y_1 \cap Y_{p+1}, \dots, \emptyset \neq Y_1 \cap Y_t.$$

The possibility $p = 1$ is allowed. Count the number of components of $Z = \pi^{-1}(Y)$. By the above, $\pi^{-1}(Y_1)$ has one component, $\pi^{-1}(Y_i)$ has n components for each $2 \leq i \leq p$, and $\pi^{-1}(Y_i)$ has one component for each $p < i \leq t$. The total number of irreducible components for $Z = \pi^{-1}(Y)$ is therefore $1 + (p - 1)n + (t - p)$.

Proposition 4.1. *With the notation established above, the sequence*

$$0 \rightarrow \left\langle \frac{f_{p+1}}{f_{p+1}^n} \right\rangle \oplus \cdots \oplus \left\langle \frac{f_t}{f_t^n} \right\rangle \xrightarrow{\beta} B(R) \rightarrow B(S) \quad (43)$$

is exact where β is “cup product with f_1 in the second factor”. That is, $\beta(u) = (u, f_1)_n$.

Proof. Clearly (43) is a complex. Since $\text{Pic } T = \text{Pic } S = 0$, the spectral sequence (19) shows $B(S/R) = H^2(G, S^*)$. Let r denote the number of irreducible components of Z . By the above, $r = 1 + (p-1)n + (t-p)$. Nagata’s theorem [7, Theorem 1.1] shows $S^*/k^* \cong \mathbb{Z}^{(r)}$.

Case 1: Suppose $p = 1$. Then $r = t$ and we easily compute

$$S^* = k^* \times \langle z \rangle \times \langle f_2 \rangle \times \cdots \langle f_t \rangle$$

where $z^n = f_1$. Writing $D = 1 - \sigma$,

$$R^* = {}_D S^* = k^* \times \langle z^n \rangle \times \langle f_2 \rangle \times \cdots \langle f_t \rangle$$

and

$$DS^* = \mu_n.$$

Setting $N = 1 + \sigma + \cdots + \sigma^{n-1}$,

$${}_N S^* = \mu_n,$$

and

$$NS^* = k^* \times \langle z^n \rangle \times \langle f_2^n \rangle \times \cdots \langle f_t^n \rangle.$$

Therefore

$$(S^*)^G = R^*,$$

and for i odd

$$H^i(G, S^*) = \mu_n / \mu_n = 1,$$

and for i even

$$H^i(G, S^*) = \frac{k^* \times \langle z \rangle \times \langle f_2 \rangle \times \cdots \langle f_t \rangle}{k^* \times \langle z^n \rangle \times \langle f_2^n \rangle \times \cdots \langle f_t^n \rangle} = \left\langle \frac{f_2}{f_2^n} \right\rangle \oplus \cdots \oplus \left\langle \frac{f_t}{f_t^n} \right\rangle$$

which shows (43) is exact in this case.

Case 2: Suppose $p \geq 2$. Again, we know $R^* = (S^*)^G$. By the decomposition of S^* , it suffices to assume $p = 2$ and $S^*/k^* = \mathbb{Z}^r = \langle f_{21} \rangle \times \cdots \times \langle f_{2n} \rangle$ and $R^* = \langle f_2 \rangle$ where $\pi^{-1}(Y_2) = Y_{21} + \cdots + Y_{2n}$. Then $N(f_{21}) = f_{21}f_{22} \cdots f_{2n} = f_2$ so

$$\frac{R^*}{NS^*} = 1$$

and we reduce back to Case 1. \square

Proposition 4.2. *Let f_1, \dots, f_t be irreducible polynomials in $k[x, y]$ defining distinct closed curves Y_1, \dots, Y_t . Let Y_0 be the line at infinity and let $X = Z(x)$. Suppose for each i that Y_i intersects $Y_0 + X$ with normal crossings. Set $R = k[x, y, x^{-1}, f_1^{-1}, \dots, f_t^{-1}]$ and $S = R[x^{1/n}]$. Then the sequence*

$$0 \rightarrow \left\langle \frac{f_1}{f_1^n} \right\rangle \oplus \cdots \oplus \left\langle \frac{f_t}{f_t^n} \right\rangle \xrightarrow{\beta} B(R) \rightarrow B(S)$$

is exact.

Proof. Proof is as in Lemma 4.1, Case 1. \square

Acknowledgement

Part of this work was carried out when the author was a visitor at the Forschungsinstitut für Mathematik, ETH, Zürich. The author is grateful for their hospitality and support.

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